THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5510 Foundation of Advanced Mathematics 2017-2018
Supplementary Exercise 1

1. Let A and B be two sets. Show that $(A\backslash B)\cap(B\backslash A)=\phi$.

   Ans:

   Note that $A\backslash B=\{x : x\in A\wedge\neg(x\in B)\}$ and $B\backslash A=\{x : x\in B\wedge\neg(x\in A)\}$, so

$$(A\backslash B)\cap(B\backslash A)=\big\{x:\big(x\in A\wedge\neg(x\in B)\big)\wedge\big(x\in B\wedge\neg(x\in A)\big)\big\}$$

   However,

$$\begin{aligned}
&\big(x\in A\wedge\neg(x\in B)\big)\wedge\big(x\in B\wedge\neg(x\in A)\big)\\
\equiv\ &\Big(\big(x\in A\wedge\neg(x\in B)\big)\wedge x\in B\Big)\wedge\neg(x\in A)\\
\equiv\ &\Big(x\in A\wedge\big(\neg(x\in B)\wedge x\in B\big)\Big)\wedge\neg(x\in A)\\
\equiv\ &(x\in A\wedge F)\wedge\neg(x\in A)\\
\equiv\ &F\wedge\neg(x\in A)\\
\equiv\ &F
\end{aligned}$$

   Therefore, $(A\backslash B)\cap(B\backslash A)=\phi$.

2. Let A and B be two sets.

   Define $A\oplus B=(A\backslash B)\cup(B\backslash A)$ which is called the symmetric difference of A and B.

   Prove that $A\oplus B=(A\cup B)\backslash(A\cap B)$.

   Ans:

$$\begin{aligned}
&\big(x\in A\wedge\neg(x\in B)\big)\vee\big(x\in B\wedge\neg(x\in A)\big)\\
\equiv\ &\Big(\big(x\in A\wedge\neg(x\in B)\big)\vee(x\in B)\Big)\wedge\Big(\big(x\in A\wedge\neg(x\in B)\big)\vee\neg(x\in A)\Big)\\
\equiv\ &\Big(\big((x\in A)\vee(x\in B)\big)\wedge\big(\neg(x\in B)\vee(x\in B)\big)\Big)\wedge\Big(\big((x\in A)\vee\neg(x\in A)\big)\wedge\big(\neg(x\in B)\vee\neg(x\in A)\big)\Big)\\
\equiv\ &\Big(\big((x\in A)\vee(x\in B)\big)\wedge T\Big)\wedge\Big(T\wedge\big(\neg(x\in B)\vee\neg(x\in A)\big)\Big)\\
\equiv\ &\big((x\in A)\vee(x\in B)\big)\wedge\big(\neg(x\in A)\vee\neg(x\in B)\big)\\
\equiv\ &\big((x\in A)\vee(x\in B)\big)\wedge\neg\big((x\in A)\wedge(x\in B)\big)
\end{aligned}$$

3. Let A, B and C be the sets of integers which are divisible by 2, 3 and 6 respectively. Prove that $A\cap B=C$.

   Ans:

   Let $m\in C$, then m is divisible by 6, i.e $m=6M$ for some $M\in\mathbb{Z}$. Then $m=2(3M)=3(2M)$, where $3M$ and $2M$ are integers. We have m is divisible by 2 and 3, which means $m\in A\cap B$. Therefore, $C\subseteq(A\cap B)$.

Let $n \in (A \cap B)$. Firstly, $n \in A$ and so $n = 2P$ for some $P \in \mathbb{Z}$. Also, $n \in B$ and so $n = 3Q$ for some $Q \in \mathbb{Z}$. Note that $n = 2P = 3Q$, since 3 is not divisible by 2, Q must be divisible by 2. Therefore, $Q = 2R$ for some integer R. Then, $n = 3Q = 6R$ which means n is divisible by 6 and so $n \in C$. Therefore, $(A \cap B) \subseteq C$.

As a result, $A \cap B = C$.

4. Let $P(x)$ and $Q(x)$ be two statement functions of x. Write down the negation of the following statements.

   (a) '$\forall x, P(x) \rightarrow Q(x)$'

   (b) '$\exists x, P(x) \wedge Q(x)$'

   Ans:

   (a) '$\exists x, P(x) \wedge (\neg Q(x))$'

   (b) '$\forall x, (\neg P(x)) \vee (\neg Q(x))$'

5. (a) Write down the negation of the following statements.

      i. For all integer n, if n is divisible by 4 then n is divisible by 2.

      ii. For all integer n, if n is divisible by 2 then n is divisible by 4.

      iii. There exists a real number x such that for all real numbers y, $xy = y$.

      iv. For all real numbers x, there exists a real number y such that $xy = 1$.

   (b) Prove or disprove the above statements.

   Ans:

   (a) i. There exists an integer n such that n is divisible by 4 but not divisible by 2.

      ii. There exists an integer n such that n is divisible by 2 but not divisible by 4.

      iii. For all real numbers x, there exists a real numbers y such that $xy \neg y$.

      iv. There exists a real number x such that for all real numbers y such that $xy \neq 1$.

   (b) i. The statement 'For all integer n, if n is divisible by 4 then n is divisible by 2.' is true. If n is an integer divisible by 4, $n = 4N$ for some integer N. Then, $n = 2(2N)$ where $2N$ is an integer. Therefore, n is divisible by 2.

      ii. The statement 'For all integer n, if n is divisible by 2 then n is divisible by 4.' is false. (To disprove it, we have to show the negation is true.) Take $n = 6$, then n is divisible by 2 but not divisible by 4.

      iii. The statement 'There exists a real number x such that for all real numbers y, $xy = y$.' is true. Take $x = 1$, then for all real numbers y, we have $xy = (1)(y) = y$

      iv. The statement 'For all real numbers x, there exists a real number y such that $xy = 1$' is false. Take $x = 0$, then for all real numbers y, we have $xy = (0)(y) = 0 \neq 1$.